Economics 150, Intermediate Microeconmics, Fall 2008

Please put your **discussion-section time** *and* **TA name** *on your answer.*

Homework 2, Wednesday, September 17
Due Wednesday, September 24

This homework examines the utility-maximization model in more detail. We first investigate a relationship that will often allow us to simplify the problem.

2.1 Suppose an individual consumes two goods, with utility function $U(x_1, x_2) = x_1^2 + 6x_1\sqrt{x_2} + 9x_2$. Formulate the utility maximization problem and find the associated first-order conditions.

2.2 Solve these first-order conditions to find the demand functions for goods 1 and 2. First divide one of the two first-order conditions by the other, so as to eliminate the Lagrange multiplier, simplify the result, and substitute into the budget constraint to find a demand function.

2.3 Now consider an individual whose utility function is given by $\tilde{U}(x_1, x_2) = x_1 + 3\sqrt{x_2}$. Formulate the utility-maximization problem, find the associated first-order conditions, and solve for the demand functions.

2.4 Compare your demand functions in parts [2.2] and [2.3]. You should find they are the same. Hence, U and $\tilde{U}$ are equivalent utility functions, in the sense that they describe the same behavior. One implication is that utility functions are not unique. Given a pattern of behavior, there are many utility functions that can be used to describe it. These equivalent utility functions are related to one another. In particular, consider the function $f(z) = z^2$. Show that $U(x_1, x_2) = f(\tilde{U}(x_1, x_2))$, i.e., $x_1^2 + 6x_1\sqrt{x_2} + 9x_2 = f(x_1 + 3\sqrt{x_2}) = (x_1 + 3\sqrt{x_2})^2$.

2.5 This relationship is general. Let us say that a function $f(z)$ is increasing if, for any z and $z' > z$, we have $f(z') > f(z)$. Following this example, define what it means for a function to be decreasing, and give an example of an increasing function, and example of a decreasing function, and an example of a function which is neither one.

2.6 Now suppose U and $\tilde{U}$ are utility functions, that f is an increasing function, and $U = f(\tilde{U})$. Write the first-order conditions for maximizing $\tilde{U}$ (subject to the usual budget constraint) and the first-order conditions for maximizing $f(\tilde{U})$. Show that if the former conditions hold, so do the latter. (You could make a similar argument for second-order conditions, but it is somewhat more involved, so do not worry about second-order conditions here.)

2.7 An implication is that if one has a complicated utility function to maximize, and can find an increasing transformation that makes it simpler, then one is free to perform the transformation. For example, if the utility functions is given by $U(x_1, x_2) = x_1^3 x_2^3$, you might as well replace it with $U(x_1, x_2) = x_1 x_2$. What is the increasing transformation involved here?

The utility function $U(x_1, x_2)$ is said to have *diminishing marginal utility* if the derivative $\frac{dU(x_1,x_2)}{dx_1}$ decreases in x_1 (and similarly for x_2, but it will suffice to concentrate on x_1).

2.8 Show that the utility function $U(x_1, x_2) = x_1^{\frac{1}{3}} x_2^{\frac{1}{3}}$ has diminishing marginal utility, the utility function $\tilde{U}(x_1, x_2) = x_1^2 x_2^2$ does not, and that the utility function $\hat{U}(x_1, x_2) = x_1 x_2$ lies right on the boundary of having diminishing returns. Show that these utility functions are increasing transformations one another. (To do this, find functions f and g such that $\tilde{U}(x_1, x_2) = f(\hat{U}(x_1, x_2)) = g(U(x_1, x_2))$.)

Since these utility functions are increasing transformations of one another, they all describe the same behavior. However, some exhibit diminishing marginal utility and some do not. The conclusion is that diminishing marginal utility is not an essential feature of a utility function. We shall see that the related notion of the marginal rate of substitution is important. Given utility function $U(x_1, x_2)$, the marginal rate of substitution at (x_1, x_2) is given by (this is sometimes defined with a negative sign in front of it)

$$\frac{\frac{dU}{dx_1}}{\frac{dU}{dx_2}}.$$

2.9 Show that the negative (this negative would be unnecessary if we had just put a negative sign in the definition of the marginal rate of substitution) of the marginal rate of substitution at (x_1, x_2) is the slope of the indifference curve through (x_1, x_2).

2.10 Show that the three utility functions in part [2.8] have the same marginal rates of substitution. Hence, the essential property of a utility function is not whether it has diminishing results, but its marginal rates of substitution. In particular, two utility functions whose marginal rates of substitution agree describe the same behavior.

2.11 Let $U(x_1, x_2)$ be a utility function and $\tilde{U}(x_1, x_2) = f(U(x_1, x_2))$ for increasing f be an increasing transformation of U. Show that these utility functions have the same marginal rates of substitution.

Consider the function $U(x_1, x_2) = x_1 + 3x_2$. Form the Lagrange function associated with maximizing this function subject to the budget constraint. To be concrete, suppose $p_1 = 1$ and $p_2 = 2$.

2.12 Find the first-order conditions for this maximization. You should encounter difficulties—it does not look as if these first-order conditions have a solution. To get some insight, draw the budget line and then some indifference curves, and find the highest indifference curve that intersects the budget line. What you are finding is that this utility maximization problem has a corner solution. One good is not consumed. Which one, and why that one? The calculus-based techniques we have worked with are appropriate for interior solutions, but not corner solutions

Consider the utility function $U(x_1, x_2) = x_1^2 + x_2^2$ that made an appearance in the previous homework.

2.13 Find the first-order conditions for this maximization. You could solve these first-order conditions, but something should trouble you about this solution. Sketch an indifference curve and a budget line (take $p_1 = p_2 = 1$ and $I = 10$), and identify the bundle (x_1, x_2) that solves your first order condition. Argue that this is a minimum of utility, subject to the constraint, rather than a maximum. To see what has gone wrong, write the second-order conditions for this maximization and show that they do not hold. From your picture, you should be able to identify a utility-maximizing bundle (there are two of them). Once again, the calculus-based techniques we have worked with are not appropriate in this case.

2.14 *Very* briefly summarize the lessons of this homework - what does the calculus-based approach to describing behavior depend on, what sorts of situations does it handle well, and when does it run into trouble?